Definitions:

- \(0 = \emptyset\).

For a set \(x\), \(S(x) = x \cup \{x\}\).

A set \(X\) is **inductive** if it contains 0 and if for all \(x \in X\), \(S(x)\) is also an element of \(X\). \(\emptyset\) is the smallest inductive set, i.e. it is the intersection of all inductive sets.

1. **Show that if \(x\) and \(y\) are sets then so is \(\{\{x\}, \{x, y\}\}\).** (7 pts.)

   **Proof:** If \(x\) and \(y\) are sets, then there is an axiom that states that \(\{x, y\}\) is a set. Taking \(x = y\), we see that \(\{x\}\) is a set as well. By the same axiom \(\{\{x\}, \{x, y\}\}\) is a set.

2. **For two sets \(x\) and \(y\), define the pair \((x, y)\) to be the set \(\{\{x\}, \{x, y\}\}\). Show that for any four sets \(x, y, z, t\), \((x, y) = (z, t)\) if and only if \(x = z\) and \(y = t\).** (7 pts.)

   **Proof:** It is clear that if \(x = z\) and \(y = t\) then \((x, y) = (z, t)\).

   Conversely, suppose that \((x, y) = (z, t)\). By definition, this means that
   \[
   \{\{x\}, \{x, y\}\} = \{\{z\}, \{z, t\}\}.
   \]

   Therefore the set \(\{x\}\) which is an element of the set \(\{\{x\}, \{x, y\}\}\) is also an element of the set \(\{\{z\}, \{z, t\}\}\). Hence either \(\{x\} = \{z\}\) or \(\{x\} = \{z, t\}\). In the first case \(x = z\) and in the second case \(z = t = x\). Thus in both cases \(x = z\). It remains to show that \(y = t\). Since
   \[
   \{\{x\}, \{x, y\}\} = \{\{z\}, \{z, t\}\}
   \]

   and since \(\{x\} = \{z\}\), we must have \(\{x, y\} = \{z, t\}\). Then the equality \(x = z\) forces the equality \(y = t\).

3. **Let \(X\) and \(Y\) be two set. Let \(Z = \wp(\wp(X \cup Y))\). Show that \((x, y) \in Z\) for all \(x \in X\) and \(y \in Y\).** (7 pts.)

   **Proof:** Note that \(\wp(\wp(X \cup Y))\) is a set by two of the axioms of set theory. Since \(x \in X\) and \(X \subseteq X \cup Y\), we have \(x \in X \cup Y\). Similarly \(y \in X \cup Y\). It follows that
   \[
   \{\{x\}\} \subseteq X \cup Y \text{ and } \{\{x, y\}\} \subseteq X \cup Y.
   \]

   Hence
   \[
   \{\{x\}\} \in \wp(X \cup Y) \text{ and } \{\{x, y\}\} \in \wp(X \cup Y).
   \]

   Therefore,
   \[
   \{\{x\}, \{x, y\}\} \subseteq \wp(X \cup Y).
   \]

   This gives
   \[
   \{\{x\}, \{x, y\}\} \in \wp(\wp(X \cup Y)) = Z.
   \]

4. **Show that the collection of all pairs \((x, y)\) for \(x \in X\) and \(y \in Y\) is a set. We denote this set by \(X \times Y\).** (7 pts.)

   **Proof:** This is the collection \(\{(x, y) \in \wp(\wp(X \cup Y)) : x \in X, y \in Y\}\). To show that this is a set we will use the third axiom of set theory given in class, namely that if \(Z\) is a set and \(\wp(z)\) is a formula, then the collection \(\{z \in Z : \wp(z)\}\) is a set.

   Let \(\alpha(x, u) = x \in u \land \forall t (t \in u \rightarrow t = x)\). Then \(\alpha(x, u)\) holds if and only if \(u = \{x\}\).

   Let \(\beta(x, y, v) = x \in v \land y \in v \land \forall t (t \in u \rightarrow (t = x \lor t = y))\). Then \(\beta(x, y, v)\) holds if and only if \(v = \{x, y\}\).
Let $\gamma(x, y, z)$ be $\exists u \exists v ((\alpha(x, u) \land \beta(x, y, v) \land \beta(u, v, z))$. Then $\gamma(x, y, z)$ holds if and only if $z = \{(x, x), (x, y)\} = (x, y)$.

Let $\varphi(z)$ be $\exists x \exists y (x \in X \land y \in Y \land \gamma(x, y, z))$. Then $\varphi(z)$ holds if and only if $z = (x, y)$ for some $x \in X$ and $y \in Y$.

Thus we have to show that $\varphi(z)$ holds if and only if $z \subseteq \rho X \cup Y) : x \in X, y \in Y}$. Then $\varphi(z)$ holds if and only if $z = (x, y)$ for some $x \in X$ and $y \in Y$.

Therefore by the axiom stated above (axiom of definable subsets or the axion of extensionality), this collection, i.e. $X \times Y$, is a set.

5. Show that the collection of all pairs $(x, y)$ such that $y = S(x)$ for some $x \in \omega$ is a subset of $\omega \times \omega$. (7 pts.)

**Proof:** We only need to show that the collection $\{(x, y) \in \omega \times \omega : y = S(x)\}$ is a set. Since we know that $\omega \times \omega$ is a set, we only need to express the condition $y = S(x)$ as a formula $\varphi(z)$.

Let $\varepsilon(x, y, z)$ be $\forall t (t \in z \leftrightarrow t \in x \lor t \in y)$. Then $\varepsilon(x, y, z)$ holds if and only if $z = x \cup y$.

Let $\psi(x, y)$ be $\exists t ((\alpha(x, t) \land \varepsilon(x, t, y))$. (Here the formula $\alpha$ is as in the previous question.) Then $\psi(x, y)$ holds if and only if $y = x \cup \{x\} = S(x)$.

Thus the collection $\{(x, y) \in \omega \times \omega : y = S(x)\}$ is also the collection $\{z \in \omega \times \omega : \exists x \exists y (\gamma(x, y, z) \land \psi(x, y))\}$, (here the formula $\gamma$ is as in the previous question) and hence is a set by the Axiom of definable sets (the famous Axiom 3).

6. Show that for all $n, m \in \omega$, if $n \neq m$ then $n \subseteq m$. (7 pts.)

**Proof:** We proceed by induction on $m$. If $m = 0$, then the statement is vacuously true. Assume the statement holds for $m$. Let $n \in S(m) = m \cup \{m\}$. Then either $n \in m$ or $n \in \{m\}$. In the first case by induction we have $n \subseteq m$; since $m \subseteq m \cup \{m\} = S(m)$, in that case we get $n \subseteq S(m)$. In the second case we must have $n = m$ and again $n = m \subseteq m \cup \{m\} = S(m)$.

7. Show that for all $n, m \in \omega$, if $S(n) = S(m)$ then either $n \in m$ or $n = m$. (7 pts.)

**Proof:** Assume $S(n) = S(m)$. Then, by definition, $n \cup \{n\} = m \cup \{m\}$. Since $n$ is an element of the set $n \cup \{n\}$, this implies that $n \in m \cup \{m\}$. Thus either $n \in m$ or $n \in \{m\}$. In the second case we get $n = m$.

8. Show that $S : \omega \rightarrow \omega$ is a one-to-one function. (7 pts.)

**Proof:** Assume that for $n, m \in \omega$, $S(n) = S(m)$ but that $n \neq m$. By question 7, either $n \in m$ or $n = m$. Therefore $n \in m$. By question 6, $n \subseteq m$. By symmetry $m \subseteq n$. Hence $n = m$.

9. Show that $S(\omega) = \omega \setminus \{0\}$. (7 pts. **Note:** Here $S(\omega)$ denotes the image of $\omega$ under the function $\omega$ and is not $\omega \cup \{\omega\}$.)

**Proof:** Since $S(n) = n \cup \{n\}$, $S(n)$ can never be empty, i.e. $S(n) \neq 0$ and so $S(\omega) \subseteq \omega \setminus \{0\}$. Conversely, we will show that for any $n \in \omega$, either $n = 0$ or $n = S(m)$ for some $m \in \omega$. We proceed by induction. If $n = 0$ the statements holds trivially. Suppose the statement holds for $n$ (we really will not care that the statement holds for $n$) and show it for $S(n)$. Thus we have to show that $S(n)$ is the $S$-image of some $m$. But $S(n)$ is of course $S$ of something, namely of $n$...
Let In the third case, We do this by induction on Assume now that We are left with the first case We will show that the same statement holds for Now we show that 


Lemma. 11. Proof: We first need a lemma.

Lemma: For all \( n, m \in \omega \) define the binary relation \( n < m \) by \( n \in m \). Show that this relation is an order on \( \omega \).

Proof: We need to show that for all \( n, m, k \in \omega \), we have

a) \( n \notin n \)
and
b) if \( n \in m \) and \( m \in k \) then \( n \in k \).
The first one is given by question 10. Assume now \( n \in m \in k \). By question 6, \( n \in m \subseteq k \).
Hence \( n \in k \).

12. Show that the order \( < \) is a total order.

Proof: We first need a lemma.

Lemma: For all \( n, m \in \omega \), if \( n \in m \) then either \( S(n) \in m \) or \( S(n) = m \).

Proof: We proceed by induction on \( m \). If \( m = 0 \) there is nothing to prove. Assume now that \( m \) is given so that the statement

For all \( n \in \omega \), if \( n \in m \) then either \( S(n) \in m \) or \( S(n) = m \)
holds (the inductive hypothesis). We will show that

For all \( n \in \omega \), if \( n \in S(m) \) then either \( S(n) \in S(m) \) or \( S(n) = S(m) \).

Let \( n \in S(m) \) be any. We will show that either \( S(n) \in S(m) \) or \( S(n) = S(m) \). Since \( n \in S(m) = m \cup \{m\} \), either \( n \in m \) or \( n = m \). In the first case, by induction, either \( S(n) \in m \) or \( S(n) = m \) and in both cases \( S(n) \in S(m) \). This proves the lemma.

Now we show that

for all \( m \in \omega \), either \( n \in m \) or \( n = m \) or \( m \in n \)
by induction on \( n \). Assume first \( n = 0 \) and choose an \( m \in \omega \). Since \( m \in 0 \) is impossible we need to show that

\[ \text{either } 0 = m \text{ or } 0 \in m. \]

We do this by induction on \( m \). If \( m = 0 \) there is nothing to prove. Assume that this holds for \( m \), we prove it for \( S(m) \). If \( m = 0 \), \( 0 = m \in S(m) \). If \( m \neq 0 \), then \( 0 \in m \subseteq S(m) \). Thus the statement is proved for \( n = 0 \).

Assume now that

for any \( m \), either \( n \in m \) or \( n = m \) or \( m \in n \).

We will show that the same statement holds for \( S(n) \) instead of \( n \), namely that

for any \( m \), either \( S(n) \in m \) or \( S(n) = m \) or \( m \in S(n) \).

Let \( m \in \omega \) be any. By induction we have three possibilities

\( n \in m \) or \( n = m \) or \( m \in n \).

In the second case, \( m = n \in S(n) \) and we are done.
In the third case, \( m \in n \subseteq S(n) \) and we are done again.
We are left with the first case \( n \in m \). But this case is dealt by the lemma above.
13. Show that any nonempty subset of $\omega$ has a least element for this order. (8 pts.)

**Proof:** Let $X$ be a nonempty subset of $\omega$. Assume that $X$ does not have a least element. We will first show by induction on $n$ that

for all $m < n$, $m \not\in X$.

If $n = 0$ this holds trivially. Assume that the statement holds for $n$. Let $m < S(n)$. Thus $m \in S(n) = n \cup \{n\}$ and either $m \in n$ or $m = n$. In the first case $m < n$ and by induction $m$ cannot be an element of $X$. In the second case, if $n$ were an element of $X$, then $n$ would be the least element of $X$ because of the inductive hypothesis; thus $n \not\in X$ either. Therefore the statement is proved. Now we show that $X = \emptyset$. Assume $n \in X$. Then $n < S(n)$ and the statement which has just been proven is false for $S(n)$, a contradiction. Therefore $X = \emptyset$.

14. Show that for any nonempty subset $X$ of $\omega$ there is an element $x \in X$ such that $x \cap X = \emptyset$. (8 pts.)

**Proof:** Let $x$ be the least element of $X$. (It exists by question 13). If $y \in x \cap X$, then $y$ would be an element of $X$ which is smaller than $x$, a contradiction. Thus $x \cap X = \emptyset$. 