Special Relativity and Kepler's First Law

Tonguç Rador[a,b*]
[a] *Boğaziçi University, Department of Physics, Bebek 34342, İstanbul, Turkey*
[b] *İzmir Institute of Technology, Department of Physics, Urla 35430, İzmir, Turkey*

We discuss a didactic example which might be helpful in undergraduate courses in relativity. The example stresses the incompatibility of the Kepler's first law with the relativistic transformation laws between two inertial observers.

Consider a two body system consisting of a very massive sun and a planet in a circular orbit of radius R around it as seen by an inertial observer $\mathcal{S}$ at rest with respect to the sun. Obviously this sun-planet system obeys Kepler's first law. For concreteness let the planet revolve on the plane $z = 0$ and that the sun is located at $x = y = z = 0$. Now consider another inertial observer $\mathcal{S}'$ moving with speed $\vec{v} = c\beta\hat{x}$, with $\beta < 1$, say on the plane $z = z_s$ relative to the sun. The Lorentz transformations relating the mentioned inertial observers are

$$ct = \gamma\,(ct' + \beta x') \tag{1a}$$
$$x = \gamma\,(x' + \beta ct') \tag{1b}$$
$$y = y' \tag{1c}$$

where $\gamma = (1-\beta^2)^{-1/2}$. Using these and the fact that in $\mathcal{S}$ the orbit obeys $x^2 + y^2 = R^2$ one can describe the orbit in terms of $\mathcal{S}'$ variables as

$$\frac{(x' + \beta ct')^2}{R^2(1-\beta^2)} + \frac{y'^2}{R^2} = 1 \tag{2}$$

As can be inferred clearly from the above, in its own inertial coordinates, the space ship will describe the planetary orbit as an ellipse that moves to the left with speed βc and with focal points located at $\pm\beta R\hat{y}$. However the sun remains at the center in clear disagreement with Kepler's first law.

The didactical merit of the example lies in its simplicity; it can be introduced right after discussing length contraction as a simple refutation of compatibility between special relativistic dynamics and laws of gravity in their Keplerian form. We can elaborate; as far as equations of motion go special relativity is a generalization of Newtonian dynamics incorporating the invariance of the speed of light. The laws in their not manifestly covariant form are the same as in Newtonian dynamics

$$\frac{d\vec{P}}{dt} = \vec{F} \tag{3a}$$
$$\frac{dE}{dt} = \vec{F}\cdot\vec{u} \tag{3b}$$

with the known relativistic definitions of momentum $\vec{P}$ and energy E. Now as is the case with Newtonian dynamics this does not tell us much about the nature[3]of $\vec{F}$; it is to be obtained via input from physical phenomena, and there is no *a priori* reason why Kepler's first law for instance should not be applicable. The example of this work if used as a homework assignment problem can also be extended by asking the direction of the force in $\mathcal{S}'$. The point of this extension would be to show that the force $\vec{F}'$ can not point towards a focal point since there are two of them and no conceivable way to choose a particular one strengthening the general argument.

We must be careful in assessing what notion this example will really provide the students with. Kepler's first law[4] is actually a statement about the orbits, not about the fundamental agent causing such orbits. Yet Kepler's statement is in principle that the orbits are conic sections. If they are closed these can be either circles or ellipses. It is evident that since Lorentz transformations are linear a closed conic curve will remain a closed conic curve. That is the statement that the orbit is either a circle or an ellipse is relativistically invariant. Thus where the law truly fails is the inclusion of the position of the sun or, for finite masses, of the position of the (Newtonian) center of mass in the statement, since the (Newtonian) center of mass is not a relativistic concept[5]. So the example we have discussed with its requirement for a massive sun is in principle equivalent to a one-body problem in a given force field but in reality it should be considered as a two-body problem to be studied fully relativistically.

Does this example even slightly point towards general relativity? Certainly not, since it does not introduce a new concept. For instance since circular geodesics are not denied in Schwarzchild[6] solution the same statement we have made about the fact that the inertial observer

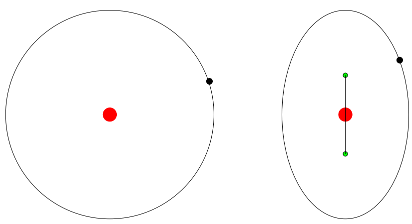

FIG. 1: The circular orbit of the planet as seen by an inertial observer at rest with respect to the sun contrasted to the orbit as seen by an inertial observer moving with$\vec{v} = c\beta\hat{x}$. The moving observer describes the orbit as an ellipse with focal points at $\pm\beta R\hat{y}$. In both pictures notice that the sun remains at the center and thus not generally at a focal point of the orbit.

$\mathcal{S}'$ describing the orbit as an ellipse is still present, but this time since general relativity is not confined to inertial observers the argument does not provide us with a criticism. The general didactical exercise in introducing general relativity, is first to point out that the main lesson to be drawn from special relativity is that every force should be represented by fields in order to circumvent immediate action at a distance and then via the use of the principle of equivalence arrive at a conclusion that the new theory should be allowing all observers, not only the inertial ones. Nevertheless even though our example can not point towards any new theory, it seems to contain a slight zest of a need to depart from special relativity if gravity is to be included in the picture. Furthermore our example does not constitute a paradox in special relativity either. It simply means what it says; Kepler's first law is incompatible with special relativity.

As a conclusion we can say that the example we have discussed may be of use to undergraduate introductory courses in special relativity. From a very simple observation it can point to various directions, philosophical and technical. In this short note we pointed out few of these.

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* tonguc.rador@boun.edu.tr

[1] Pryce, M.H.L., Proc. Roy. Soc. Lon. 195, 62-81, 1948.

[2] Chryssomalakos C., Hernandez-Coronado H. and Okon E., Journal of Physics: Conference Series 174(2009), 012026.

[3] There is however a strict condition on the Minkowski four-force M^μ containing $\vec{F}$. Since $P_\mu P^\mu$ is both an invariant and a constant of motion one can easily find that $M^\mu P_\mu = 0$ which is most naturally achieved if $M^\mu = F^{\mu\nu} P_\nu$ where F is an antisymmetric field strength. Such a constraint is absent in Newtonian dynamics and the most one can obtain for interactions between two bodies is that the force depends on the difference of their position vectors as a result of the observation that this difference is invariant under Gallilean transformations.

[4] First called a *law* by Voltaire in "Elements of Newton's Philosophy", 1738.

[5] In fact even a relativistic generalization of the concept of center of mass as for instance done by Pryce[1] yields non-commuting poisson brackets of its coordinates and thus is not free of obstacles. See also[2] and the references therein.

[6] However there is a lower bound below which circular orbits are unstable.