QUESTIONS

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(1) Show that there exist arbitrarily large intervals that are free of primes, i.e., for every positive integer k there exist k consecutive positive integers none of which is a prime.
(2) Prove that $\mu(n)\mu(n+1)\mu(n+2)\mu(n+3) = 0$ if n is a positive integer.
(3) Find all n such that $\phi(n) = 12$ and $n = 17\phi(n)$
(4) Find a positive integer n such that $\mu(n) + \mu(n+1) + \mu(n+2) = 3$.
(5) Show that $H_n = 1 + 1/2 + ... + 1/n$ is not an integer for $n \geq 2$.
(6) Show that, for every positive integer $n \geq 2$,

$$\sum_{\substack{1\leq k\leq n-1\\(k,n)=1}} k = \frac{n}{2}\phi(n)$$

(7) Let f be a multiplicative function. We know that the Dirichlet inverse f^{-1} is then also multiplicative. Show that f^{-1} is completely multiplicative if and only if $f(p^m) = 0$ for all prime powers p^m with $m \geq 2$.
(8) If n is any even integer, prove that $\sum_{d/n} \mu(d)\phi(d) = 0$.
(9) Let f_k is defined as follows

$$f_k(n) = \sum_{\substack{d/n\\(k,d)=1}} \mu(d),$$

here k is a fixed positive integer, and the summation runs over those divisors of n that are relatively prime to k. Show that f_k is the characteristic function of the set $A_k = \{n \in \mathbb{N}\colon p/n \Longrightarrow p/k\}$
(10) Show that $\exp(\frac{\log x}{\log\log x}) = o(x^\epsilon)$ for any $\epsilon > 0$.
(11) For any positive integer n, prove that $\phi(n) + \sigma(n) \geq 2n$ and the equality holds iff $n = 1$ or prime.
(12) Show that $\psi(x) = \theta(x) + O(\sqrt{x})$.
(13) Let $\omega(n)$ be the number of distinct prime factors of n. Show that $\omega(n) \leq 2\log n$.
(14) Let f be a multiplicative function and suppose that $\lim_{p^m\to\infty} f(p^m) = 0$. Show that $\lim_{n\to\infty} f(n) = 0$ also.
(15) Show that $\frac{n}{\log n} << \phi(n)$ for $n \geq 2$.
(16) Let $d(n)$ be the number of divisors of n. Show that $d(n) = O(n^\epsilon)$ for every $\epsilon > 0$.
(17) Show that $d(n) = O(\log n)$ is not true.
(18) Show that $\frac{n}{\phi(n)} = \sum_{d|n} \frac{\mu(n)^2}{\phi(n)}$
(19) Show that $\sum_{n\leq x} \frac{\mu(n)^2}{\phi(n)} \geq \log x$.
(20) Show that $\sum_{n\leq x} \frac{n}{\phi(n)} << x$. Moreover show that for any fixed real number k, $\sum_{n\leq x}(\frac{n}{\phi(n)})^k << x$. This means most of the time $\phi(n)$ is very close to n.
(21) Let P be a set of primes such that $\sum_{p\in P}\frac{1}{p}$ is finite.
Define $A_P = \{n : p|n \to p \in P\}$, $A_P(x) = |\{n \leq x : n \in A_P\}|$,

$C = \{n : (n,p) = 1 \forall p \in P\}$. Show that $\sum_{n \in A_P} \frac{1}{n}$ is finite, $A_P(x) = o(x)$, and $C(x)$ is asymptotic to ax where $a = \prod_{p \in P}(1 - 1/p)$.

(22) Show that $|\{n \leq x : p|n \to p = 4k+1\}| = o(x)$.

(23) Show that $\pi_3(x) << \frac{x}{\log^3 x}$ where $\pi_3(x)$=the number of primes $p \leq x$ such that $p+2$ and $p+6$ are also primes.

(24) Show that $\sum_{p \leq x} d(p-1) = O(x)$.

(25) Show that primes of the form n^2+1 with $n \leq x$ is $<< \frac{x}{\log x}$.

(26) Prove that Selberg's asymptotic formula
$\psi(x)\log x + \sum_{n \leq x} \Lambda(n)\psi(\frac{x}{n}) = 2x\log x + O(x)$ implies Chebyshev estimates. (In fact Selberg's formula has a key role for the elementary proof of PNT.)

(27) Using Merten's estimates find the asymptotic of $\sum_{pq \leq x} \frac{1}{pq}$. Using PNT prove that $\sum_{pq \leq x} 1$ is asymptotic to $\frac{x \log\log x}{\log x}$.

(28) It is known that PNT is equivalent to $\sum_{n \leq x} \mu(n) = o(x)$. Using this show that PNT iff $\lim_{x \to \infty} x \sum_{n > x} \frac{\mu(n)}{n^2} = 0$

(29) Let $E = \liminf_n \frac{p_{n+1} - p_n}{\log p_n}$. Using PNT show that $E \leq 1$.

(30) Using Brun's sieve what can you say about a lower bound for $\pi(x)$ and $\pi_2(x)$?