BORELSETS, WELL-ORDERINGS OF $\mathbb{R}$ AND THE CONTINUUM HYPOTHESIS

SIMON THOMAS

1. The Finite Basis Problem

Definition 1.1. Let $\mathcal{C}$ be a class of structures. Then a *basis* for $\mathcal{C}$ is a collection $\mathcal{B} \subseteq \mathcal{C}$ such that for every $C \in \mathcal{C}$, there exists $B \in \mathcal{B}$ such that B embeds into C.

Theorem 1.2 (Ramsey). *If $\chi : [\mathbb{N}]^2 \to 2$ is any function, then there exists an infinite $X \subseteq \mathbb{N}$ such that $\chi \restriction [X]^2$ is a constant function.*

Proof. We shall define inductively a decreasing sequence of infinite subsets of $\mathbb{N}$

$$\mathbb{N} = S_0 \supset S_1 \supset S_2 \supset \cdots \supset S_n \supset \cdots$$

together with an associated increasing sequence of natural numbers

$$0 = a_0 < a_1 < a_2 < \cdots < a_n < \cdots$$

with $a_n = \min S_n$ as follows. Suppose that S_n has been defined. For each $\varepsilon = 0, 1$, define

$$S_n^\varepsilon = \{\ell \in S_n \smallsetminus \{a_n\} \mid \chi(\{a_n, \ell\}) = \varepsilon\}.$$

Then we set

$$S_{n+1} = \begin{cases} S_n^0, & \text{if } S_n^0 \text{ is infinite;} \\ S_n^1, & \text{otherwise.} \end{cases}$$

Notice that if $n < m < \ell$, then a_m, $a_\ell \in S_{n+1}$ and so

$$\chi(\{a_n, a_m\}) = \chi(\{a_n, a_\ell\}).$$

Thus there exists $\varepsilon_n \in 2$ such that

$$\chi(\{a_n, a_m\}) = \varepsilon_n \quad \text{for all } m > n.$$

There exists a fixed $\varepsilon \in 2$ and an infinite $E \subseteq \mathbb{N}$ such that $\varepsilon_n = \varepsilon$ for all $n \in E$. Hence $X = \{a_n \mid n \in E\}$ satisfies our requirements. $\square$

Corollary 1.3. *Each of the following classes has a finite basis:*

(i) *the class of countably infinite graphs;*
(ii) *the class of countably infinite linear orders;*
(iiI) *the class of countably infinite partial orders.*

Example 1.4. The class of countably infinite groups does *not* admit a countable basis.

Theorem 1.5 (Sierpinski)**.** $\omega_1, \omega_1^* \not\hookrightarrow \mathbb{R}$.

Proof. Suppose that $f : \omega_1 \hookrightarrow \mathbb{R}$ is order-preserving. If $\operatorname{ran} f$ is bounded above, then it has a least upper bound $r \in \mathbb{R}$. Hence, since $(-\infty, r) \cong \mathbb{R}$, we can suppose that $\operatorname{ran} f$ is unbounded in $\mathbb{R}$. Then for each $n \in \mathbb{N}$, there exists $\alpha_n \in \omega_1$ such that $f(\alpha_n) > n$. Hence if $\alpha = \sup \alpha_n \in \omega_1$, then $f(\alpha) > n$ for all $n \in \mathbb{N}$, which is a contradiction. □

Theorem 1.6 (Sierpinski)**.** *There exists an uncountable graph* $\Gamma = \langle \mathbb{R}, E \rangle$ *such that:*

- Γ *does not contain an uncountable complete subgraph.*
- Γ *does not contain an uncountable null subgraph.*

Proof. Let $\prec$ be a well-ordering of $\mathbb{R}$ and let $<$ be the usual ordering. If $r \neq s \in \mathbb{R}$, then we define

$$r \, E \, s \qquad \text{iff} \qquad r < s \Longleftrightarrow r \prec s.$$

□

Question 1.7. Can you find an *explicit* well-ordering of $\mathbb{R}$?

Question 1.8. Can you find an *explicit* example of a subset $A \subseteq \mathbb{R}$ such that $|A| = \aleph_1$?

An Analogue of Church's Thesis. *The explicit subsets of* $\mathbb{R}^n$ *are precisely the Borel subsets.*

Definition 1.9. The collection $\mathbf{B}(\mathbb{R}^n)$ of *Borel subsets* of $\mathbb{R}^n$ is the smallest collection such that:

(a) If $U \subseteq \mathbb{R}^n$ is open, then $U \in \mathbf{B}(\mathbb{R}^n)$.

(b) If $A \in \mathbf{B}(\mathbb{R}^n)$, then $\mathbb{R}^n \smallsetminus A \in \mathbf{B}(\mathbb{R}^n)$.

(c) If $A_n \in \mathbf{B}(\mathbb{R}^n)$ for each $n \in \mathbb{N}$, then $\bigcup A_n \in \mathbf{B}(\mathbb{R}^n)$.

In other words, $\mathbf{B}(\mathbb{R}^n)$ is the *σ-algebra* generated by the collection of open subsets of $\mathbb{R}^n$.

Main Theorem 1.10. *If $A \subseteq \mathbb{R}$ is a Borel subset, then either A is countable or else $|A| = |\mathbb{R}|$.*

Definition 1.11. A binary relation R on $\mathbb{R}$ is said to be *Borel* iff R is a Borel subset of $\mathbb{R} \times \mathbb{R}$.

Example 1.12. The usual order relation on $\mathbb{R}$

$$R = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid x < y\}$$

is an open subset of $\mathbb{R} \times \mathbb{R}$. Hence R is a Borel relation.

Main Theorem 1.13. *There does* not *exist a Borel well-ordering of* $\mathbb{R}$.

2. Topological Spaces

Definition 2.1. If (X, d) is a metric space, then the induced topological space is $(X, \mathcal{T})$, where $\mathcal{T}$ is the topology with open basis

$$B(x, r) = \{y \in X \mid d(x, y) < r\} \quad x \in X, r > 0.$$

In this case, we say that the metric d is *compatible* with the topology $\mathcal{T}$ and we also say that the topology $\mathcal{T}$ is *metrizable*.

Definition 2.2. A topological space X is said to be *Hausdorff* iff for all $x \neq y \in X$, there exist disjoint open subsets U, $V \subseteq X$ such that $x \in U$ and $y \in V$.

Remark 2.3. If X is a metrizable space, then X is Hausdorff.

Definition 2.4. Let X be a Hausdorff space. If $(a_n)_{n \in \mathbb{N}}$ is a sequence of elements of X and $b \in X$, then $\lim a_n = b$ iff for every open nbhd U of b, we have that $a_n \in U$ for all but finitely many n.

Definition 2.5. If X, Y are topological spaces, then the map $f : X \to Y$ is *continuous* iff whenever $U \subseteq Y$ is open, then $f^{-1}(U) \subseteq X$ is also open.

Definition 2.6. Let $(X, \mathcal{T})$ be a topological space. Then the collection $\mathbf{B}(\mathcal{T})$ of *Borel subsets* of X is the smallest collection such that:

(a) $\mathcal{T} \subseteq \mathbf{B}(\mathcal{T})$.
(b) If $A \in \mathbf{B}(\mathcal{T})$, then $X \smallsetminus A \in \mathbf{B}(\mathcal{T})$.
(c) If $A_n \in \mathbf{B}(\mathcal{T})$ for each $n \in \mathbb{N}$, then $\bigcup A_n \in \mathbf{B}(\mathcal{T})$.

In other words, $\mathbf{B}(\mathcal{T})$ is the σ*-algebra* generated by $\mathcal{T}$. We sometimes write $\mathbf{B}(X)$ instead of $\mathbf{B}(\mathcal{T})$.

Example 2.7. Let d be the usual Euclidean metric on $\mathbb{R}^2$ and let $(\mathbb{R}^2, \mathcal{T})$ be the corresponding topological space. Then the *New York metric*

$$\hat{d}(\bar{x}, \bar{y}) = |x_1 - y_1| + |x_2 - y_2|$$

is also compatible with $\mathcal{T}$.

Remark 2.8. Let $(X, \mathcal{T})$ be a metrizable space and let d be a compatible metric. Then

$$\hat{d}(x, y) = \min\{d(x, y), 1\}$$

is also a compatible metric.

Definition 2.9. A metric (X, d) is *complete* iff every Cauchy sequence converges.

Example 2.10. The usual metric on $\mathbb{R}^n$ is complete. Hence if $C \subseteq \mathbb{R}^n$ is closed, then the metric on C is also complete.

Example 2.11. If X is any set, the *discrete metric* on X is defined by

$$d(x, y) = \begin{cases} 0, & \text{if } x = y; \\ 1, & \text{otherwise.} \end{cases}$$

Clearly the discrete metric is complete.

Definition 2.12. Let $(X, \mathcal{T})$ be a topological space.

(a) $(X, \mathcal{T})$ is *separable* iff it has a countable dense subset.
(b) $(X, \mathcal{T})$ is a *Polish space* iff it is separable and there exists a compatible complete metric d.

Example 2.13. Let $2^{\mathbb{N}}$ be the set of all infinite binary sequences

$$(a_n) = (a_0, a_1, \cdots, a_n, \cdots),$$

where each $a_n = 0$, 1. Then we can define a metric on $2^{\mathbb{N}}$ by

$$d((a_n), (b_n)) = \sum_{n=0}^{\infty} \frac{|a_n - b_n|}{2^{n+1}}.$$

The corresponding topological space $(2^{\mathbb{N}}, \mathcal{T})$ is called the *Cantor space*. It is easily checked that $2^{\mathbb{N}}$ is a Polish space. For each finite sequence $\bar{c} = (c_0, \cdots, c_\ell) \in 2^{<\mathbb{N}}$, let

$$U_{\bar{c}} = \{(a_n) \in 2^{\mathbb{N}} \mid a_n = c_n \text{ for all } 0 \leq n \leq \ell\}.$$

Then $\{U_{\bar{c}} \mid \bar{c} \in 2^{<\mathbb{N}}\}$ is a countable basis of open sets.

Remark 2.14. Let $(X, \mathcal{T})$ be a separable metrizable space and let d be a compatible metric. If $\{x_n\}$ is a countable dense subset, then

$$B(x_n, 1/m) = \{y \in X \mid d(x_n, y) < 1/m\} \quad n \in \mathbb{N}, 0 < m \in \mathbb{N},$$

is a countable basis of open sets.

Example 2.15 (The Sorgenfrey Line). Let $\mathcal{T}$ be the topology on $\mathbb{R}$ with basis

$$\{\, [r, s) \mid r < s \in \mathbb{R} \,\}.$$

Then $(X, \mathcal{T})$ is separable but does *not* have a countable basis of open sets.

Definition 2.16. If (X_1, d_1) and (X_2, d_2) are metric spaces, then the *product metric* on $X_1 \times X_2$ is defined by

$$d(\bar{x}, \bar{y}) = d_1(x_1, y_1) + d_2(x_2, y_2).$$

The corresponding topology has an open basis

$$\{U_1 \times U_2 \mid U_1 \subseteq X_1 \text{ and } U_2 \subseteq X_2 \text{ are open }\}.$$

Definition 2.17. For each $n \in \mathbb{N}$, let (X_n, d_n) be a metric space. Then the *product metric* on $\prod_n X_n$ is defined by

$$d(\bar{x}, \bar{y}) = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} \min\{d_n(x_n, y_n), 1\}.$$

The corresponding topology has an open basis consisting of sets of the form

$$U_0 \times U_1 \times \cdots \times U_n \times \cdots,$$

where each $U_n \subseteq X_n$ is open and $U_n = X_n$ for all but finitely many n.

Example 2.18. The Cantor space $2^{\mathbb{N}}$ is the product of countably many copies of the discrete space $2 = \{0, 1\}$.

Theorem 2.19. *If X_n, $n \in \mathbb{N}$, are Polish spaces, then $\prod_n X_n$ is also Polish.*

Proof. For example, to see that $\prod_n X_n$ is separable, let $\{V_{n,\ell} \mid \ell \in \mathbb{N}\}$ be a countable open basis of X_n for each $n \in \mathbb{N}$. Then $\prod_n X_n$ has a countable open basis consisting of the sets of the form

$$U_0 \times U_1 \times \cdots \times U_n \times \cdots,$$

where each $U_n \in \{V_{n,\ell} \mid \ell \in \mathbb{N}\} \cup \{X_n\}$ and $U_n = X_n$ for all but finitely many n. Choosing a point in each such open set, we obtain a countable dense subset. $\square$

3. Perfect Polish Spaces

Definition 3.1. A topological space X is *compact* iff whenever $X = \bigcup_{i \in I} U_i$ is an open cover, there exists a finite subset $I_0 \subseteq I$ such that $X = \bigcup_{i \in I_0} U_i$.

Remark 3.2. If (X, d) is a metric space, then the topological space $(X, \mathcal{T})$ is compact iff every sequence has a convergent subsequence.

Theorem 3.3. *The Cantor space is compact.*

Definition 3.4. If $(X, \mathcal{T})$ is a topological space and $Y \subseteq X$, then the *subspace topology* on Y is $\mathcal{T}_Y = \{ Y \cap U \mid U \in \mathcal{T} \}$.

Theorem 3.5. (a) *A closed subset of a compact space is compact.*

(b) *Suppose that $f : X \to Y$ is a continuous map between the topological spaces X, Y. If $Z \subseteq X$ is compact, then $f(Z)$ is also compact.*

(c) *Compact subspaces of Hausdorff spaces are closed.*

Definition 3.6. Let X be a topological space.

(i) The point x is a *limit point* of X iff $\{x\}$ is *not* open.

(ii) X is *perfect* iff all its points are limit points.

(iii) $Y \subseteq X$ is a *perfect subset* iff Y is closed and perfect in its subspace topology.

Theorem 3.7. *If X is a nonempty perfect Polish space, then there is an embedding of the Cantor set $2^{\mathbb{N}}$ into X.*

Definition 3.8. A map $f : X \to Y$ between topological spaces is an *embedding* iff f induces a homeomorphism between X and $f(X)$. (Here $f(X)$ is given the subspace topology.)

Lemma 3.9. *A continuous injection $f : X \to Y$ from a compact space into a Hausdorff space is an embedding.*

Proof. It is enough to show that if $U \subseteq X$ is open, then $f(U)$ is open in $f(X)$. Since $X \smallsetminus U$ is closed and hence compact, it follows that $f(X \smallsetminus U)$ is compact in Y. Since Y is Hausdorff, it follows that $f(X \smallsetminus U)$ is closed in Y. Hence

$$f(U) = (\, Y \smallsetminus f(X \smallsetminus U)\,) \cap f(X)$$

is an open subset of $f(X)$. □

Definition 3.10. A *Cantor scheme* on a set X is a family $(A_s)_{s \in 2^{<\mathbb{N}}}$ of subsets of X such that:

(i) $A_{s^\frown 0} \cap A_{s^\frown 1} = \emptyset$ for all $s \in 2^{<\mathbb{N}}$.

(ii) $A_{s^\frown i} \subseteq A_s$ for all $s \in 2^{<\mathbb{N}}$ and $i \in 2$.

Proof of Theorem 3.7. Let d be a complete compatible metric on X. We will define a Cantor scheme $(U_s)_{s \in 2^{<\mathbb{N}}}$ on X such that:

(a) U_s is a nonempty open ball;

(b) $\operatorname{diam}(U_s) \leq 2^{-\operatorname{length}(s)}$;

(c) $\operatorname{cl}(U_{s^\frown i}) \subseteq U_s$ for all $s \in 2^{<\mathbb{N}}$ and $i \in 2$.

Then for each $\varphi \in 2^{\mathbb{N}}$, we have that $\bigcap U_{\varphi \restriction n} = \bigcap \operatorname{cl}(U_{\varphi \restriction n})$ is a singleton; say $\{f(\varphi)\}$. Clearly the map $f : 2^{\mathbb{N}} \to X$ is injective and continuous, and hence is an embedding.

We define U_s by induction on $\operatorname{length}(s)$. Let $U_\emptyset$ be an arbitrary nonempty open ball with $\operatorname{diam}(U_\emptyset) \leq 1$. Given U_s, choose $x \neq y \in U_s$ and let $U_{s^\frown 0}$, $U_{s^\frown 1}$ be sufficiently small open balls around x, y respectively. □

Definition 3.11. A point x in a topological space X is a *condensation point* iff every open nbhd of x is uncountable.

Theorem 3.12 (Cantor-Bendixson Theorem)**.** *If X is a Polish space, then X can be written as $X = P \cup C$, where P is a perfect subset and C is a countable open subset.*

Proof. Let $P = \{x \in X \mid x \text{ is a condensation point of } X\}$ and let $C = X \smallsetminus P$. Let $\{U_n\}$ be a countable open basis of X. Then $C = \bigcup\{U_n \mid U_n \text{ is countable }\}$ and hence C is a countable open subset. To see that P is perfect, let $x \in P$ and let U be an open nbhd of x in X. Then U is uncountable and hence $U \cap P$ is also uncountable. □

Corollary 3.13. *Any uncountable Polish space contains a homeomorphic copy of the Cantor set* $2^{\mathbb{N}}$.

4. Polish subspaces

Theorem 4.1. *If X is a Polish space and $U \subseteq X$ is open, then U is a Polish subspace.*

Proof. Let d be a complete compatible metric on X. Then we can define a metric $\hat{d}$ on U by

$$\hat{d}(x,y) = d(x,y) + \left| \frac{1}{d(x, X \smallsetminus U)} - \frac{1}{d(y, X \smallsetminus U)} \right|.$$

It is easily checked that $\hat{d}$ is a metric. Since $\hat{d}(x,y) \geq d(x,y)$, every d-open set is also $\hat{d}$-open. Conversely suppose that $x \in U$, $d(x, X \smallsetminus U) = r > 0$ and $\varepsilon > 0$. Choose $\delta > 0$ such that if $0 < \eta \leq \delta$, then $\eta + \frac{\eta}{r(r-\eta)} < \varepsilon$. If $d(x,y) = \eta < \delta$, then $r - \eta \leq d(y, X \smallsetminus U) \leq r + \eta$ and hence

$$\frac{1}{r} - \frac{1}{r-\eta} \leq \frac{1}{d(x, X \smallsetminus U)} - \frac{1}{d(y, X \smallsetminus U)} \leq \frac{1}{r} - \frac{1}{r+\eta}$$

and so

$$\frac{-\eta}{r(r-\eta)} \leq \frac{1}{d(x, X \smallsetminus U)} - \frac{1}{d(y, X \smallsetminus U)} \leq \frac{\eta}{r(r+\eta)}.$$

Thus $\hat{d}(x,y) \leq \eta + + \frac{\eta}{r(r-\eta)} < \varepsilon$. Thus the $\hat{d}$-ball of radius ε around x contains the d-ball of radius δ and so every $\hat{d}$-open set is also d-open. Thus $\hat{d}$ is compatible with the subspace topology on U and we need only show that $\hat{d}$ is complete.

Suppose that (x_n) is a $\hat{d}$-Cauchy sequence. Then (x_n) is also a d-Cauchy sequence and so there exists $x \in X$ such that $x_n \to x$. In addition,

$$\lim_{i,j\to\infty} \left| \frac{1}{d(x_i, X \smallsetminus U)} - \frac{1}{d(x_j, X \smallsetminus U)} \right| = 0$$

and so there exists $s \in \mathbb{R}$ such that

$$\lim_{i\to\infty} \frac{1}{d(x_i, X \smallsetminus U)} = s.$$

In particular, $d(x_i, X \smallsetminus U)$ is bounded away from 0 and hence $x \in U$. □

Definition 4.2. A subset Y of a topological space is said to be a G_δ*-set* iff there exist open subsets $\{V_n\}$ such that $Y = \bigcap V_n$.

Example 4.3. Suppose that X is a metrizable space and that d is a compatible metric. If $F \subseteq X$ is closed, then

$$F = \bigcap_{n=1}^{\infty} \{x \in X \mid d(x, F) < 1/n\}$$

is a G_δ-set.

Corollary 4.4. *If X is a Polish space and $Y \subseteq X$ is a G_δ-set, then Y is a Polish subspace.*

Proof. Let $Y = \bigcap V_n$, where each V_n is open. By Theorem 4.1, each V_n is Polish. Let d_n be a complete compatible metric on V_n such that $d_n \leq 1$. Then we can define a complete compatible metric on Y by

$$\hat{d}(x, y) = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} d_n(x, y).$$

The details are left as an exercise for the reader. □

Example 4.5. Note that $\mathbb{Q} \subseteq \mathbb{R}$ is *not* a Polish subspace.

Theorem 4.6. *If X a Polish space and $Y \subseteq X$, then Y is a Polish subspace iff Y is a G_δ-set.*

Proof. Suppose that Y is a Polish subspace and let d be a complete compatible metric on Y. Let $\{U_n\}$ be an open basis for X. Then for every $y \in Y$ and $\varepsilon > 0$, there exists U_n such that $y \in U_n$ and $\operatorname{diam}(Y \cap U_n) < \varepsilon$, where the diameter is computed with respect to d. Let

$$\begin{aligned} A &= \{x \in \operatorname{cl}(Y) \mid (\forall \varepsilon > 0)\,(\exists n)\ x \in U_n \text{ and } \operatorname{diam}(Y \cap U_n) < \varepsilon\} \\ &= \bigcap_{m=1}^{\infty} \bigcup \{U_n \cap \operatorname{cl}(Y) \mid \operatorname{diam}(Y \cap U_n) < 1/m\}. \end{aligned}$$

Thus A is a G_δ-set in $\mathrm{cl}(Y)$. Since $\mathrm{cl}(Y)$ is a G_δ-set in X, it follows that A is a G_δ-set in X. Furthermore, we have already seen that $Y \subseteq A$.

Suppose that $x \in A$. Then for each $m \geq 1$, there exists U_{n_m} such that $x \in U_{n_m}$ and $\mathrm{diam}(Y \cap U_{n_m}) < 1/m$. Since Y is dense in A, for each $m \geq 1$, there exists $y_m \in Y \cap U_{n_1} \cap \cdots \cap U_{n_m}$. Thus y_1, y_2, ... is a d-Cauchy sequence which converges to x and so $x \in Y$. Thus $Y = A$ is a G_δ-set. □

5. Changing The Topology

Theorem 5.1. *Let $(X, \mathcal{T})$ be a Polish space and let $A \subseteq X$ be a Borel subset. Then there exists a Polish topology $\mathcal{T}_A \supseteq \mathcal{T}$ on X such that $\mathbf{B}(\mathcal{T}) = \mathbf{B}(\mathcal{T}_A)$ and A is clopen in $(X, \mathcal{T}_A)$.*

Theorem 5.2 (The Perfect Subset Theorem)**.** *Let X be a Polish space and let $A \subseteq X$ be an uncountable Borel subset. Then A contains a homeomorphic copy of the Cantor set $2^{\mathbb{N}}$.*

Proof. Extend the topology $\mathcal{T}$ of X to a Polish topology $\mathcal{T}_A$ with $\mathbf{B}(\mathcal{T}) = \mathbf{B}(\mathcal{T}_A)$ such that A is clopen in $(X, \mathcal{T}_A)$. Equipped with the subspace topology $\mathcal{T}_A'$ relative to $(X, \mathcal{T}_A)$, we have that $(A, \mathcal{T}_A')$ is an uncountable Polish space. Hence there exists an embedding $f : 2^{\mathbb{N}} \to (A, \mathcal{T}_A')$. Clearly f is also a continuous injection of $2^{\mathbb{N}}$ into $(X, \mathcal{T}_A)$ and hence also of $2^{\mathbb{N}}$ into $(X, \mathcal{T})$. Since $2^{\mathbb{N}}$ is compact, it follows that f is an embedding of $2^{\mathbb{N}}$ into $(X, \mathcal{T})$. □

We now begin the proof of Theorem 5.1.

Lemma 5.3. *Suppose that $(X_1, \mathcal{T}_1)$ and $(X_2, \mathcal{T}_2)$ are disjoint Polish spaces. Then the disjoint union $(X_1 \sqcup X_2, \mathcal{T})$, where $\mathcal{T} = \{U \sqcup V \mid U \in \mathcal{T}_1, V \in \mathcal{T}_2\}$, is also a Polish space.*

Proof. Let d_1, d_2 be compatible complete metrics on X_1, X_2 such that d_1, $d_2 \leq 1$. Let $\hat{d}$ be the metric defined on $X_1 \sqcup X_2$ by

$$\hat{d}(x, y) = \begin{cases} d_1(x, y), & \text{if } x,\ y \in X_1; \\ d_2(x, y), & \text{if } x,\ y \in X_2; \\ 2, & \text{otherwise.} \end{cases}$$

Then $\hat{d}$ is a complete metric which is compatible with $\mathcal{T}$. □

Lemma 5.4. *Let $(X, \mathcal{T})$ be a Polish space and let $F \subseteq X$ be a closed subset. Let $\mathcal{T}_F$ be the topology generated by $\mathcal{T} \cup \{F\}$. Then $(X, \mathcal{T}_F)$ is a Polish space, F is clopen in $(X, \mathcal{T}_F)$, and $\mathbf{B}(\mathcal{T}) = \mathbf{B}(\mathcal{T}_F)$.*

Proof. Clearly $\mathcal{T}_F$ is the topology with open basis $\mathcal{T} \cup \{U \cap F \mid U \in \mathcal{T}\}$ and so $\mathcal{T}_F$ is the disjoint union of the relative topologies on $X \smallsetminus F$ and F. Since F is closed and $X \smallsetminus F$ is open, it follows that their relatives topologies are Polish. So the result follows by Lemma 5.3. □

Lemma 5.5. *Let $(X, \mathcal{T})$ be a Polish space and let $(\mathcal{T}_n)$ be a sequence of Polish topologies on X such that $\mathcal{T} \subseteq \mathcal{T}_n \subseteq \mathbf{B}(\mathcal{T})$ for each $n \in \mathbb{N}$. Then the topology $\mathcal{T}_\infty$ generated by $\bigcup \mathcal{T}_n$ is Polish and $\mathbf{B}(\mathcal{T}) = \mathbf{B}(\mathcal{T}_\infty)$.*

Proof. For each $n \in \mathbb{N}$, let X_n denote the Polish space $(X, \mathcal{T}_n)$. Consider the diagonal map $\varphi : X \to \prod X_n$ defined by $\varphi(x) = (x, x, x, \cdots)$. We claim that $\varphi(X)$ is closed in $\prod X_n$. To see this, suppose that $(x_n) \notin \varphi(X)$; say, $x_i \neq x_j$. Then there exist disjoint open sets U, $V \in \mathcal{T} \subseteq \mathcal{T}_i$, $\mathcal{T}_j$ such that $x_i \in U$ and $x_j \in V$. Then

$$(x_n) \in X_0 \times \cdots \times X_{i-1} \times U \times X_{i+1} \times \cdots \times X_{j-1} \times V \times X_{j+1} \times \cdots \subseteq \prod X_n \smallsetminus \varphi(X).$$

In particular, $\varphi(X)$ is a Polish subspace of $\prod X_n$; and it is easily checked that φ is a homeomorphism between $(X, \mathcal{T}_\infty)$ and $\varphi(X)$. □

Proof of Theorem 5.1. Consider the class

$$\mathcal{S} = \{A \in \mathbf{B}(\mathcal{T}) \mid A \text{ satisfies the conclusion of Theorem 5.1 } \}.$$

It is enough to show that $\mathcal{S}$ is a σ-algebra such that $\mathcal{T} \subseteq \mathcal{S}$. Clearly $\mathcal{S}$ is closed under taking complements. In particular, Lemma 5.4 implies that $\mathcal{T} \subseteq \mathcal{S}$. Finally suppose that $\{A_n\} \subseteq \mathcal{S}$. For each $n \in \mathbb{N}$, let $\mathcal{T}_n$ be a Polish topology which witnesses that $A_n \in \mathcal{S}$ and let $\mathcal{T}_\infty$ be the Polish topology generated by $\bigcup \mathcal{T}_n$. Then $A = \bigcup A_n$ is open in $\mathcal{T}_\infty$. Applying Lemma 5.4 once again, there exists a Polish topology $\mathcal{T}_A \supseteq \mathcal{T}_\infty$ such that $\mathbf{B}(\mathcal{T}_A) = \mathbf{B}(\mathcal{T}_\infty) = \mathbf{B}(\mathcal{T})$ and A is clopen in $(X, \mathcal{T}_A)$. Thus $A \in \mathcal{S}$. □

6. The Borel Isomorphism Theorem

Definition 6.1. If $(X, \mathcal{T})$ is a topological space, then the corresponding *Borel space* is $(X, \mathbf{B}(\mathcal{T}))$.

Theorem 6.2. *If $(X, \mathcal{T})$ and $(Y, \mathcal{S})$ are uncountable Polish spaces, then the corresponding Borel spaces $(X, \mathbf{B}(\mathcal{T}))$ and $(Y, \mathbf{B}(\mathcal{S}))$ are isomorphic.*

Definition 6.3. Let $(X, \mathcal{T})$ and $(Y, \mathcal{S})$ be topological spaces and let $f : X \to Y$.

(a) f is a *Borel map* iff $f^{-1}(A) \in \mathbf{B}(\mathcal{T})$ for all $A \in \mathbf{B}(\mathcal{S})$.

(b) f is a *Borel isomorphism* iff f is a Borel bijection such that f^{-1} is also a Borel map.

Definition 6.4. Let $(X, \mathcal{T})$ be a topological space and let $Y \subseteq X$. Then the *Borel subspace structure* on Y is defined to be $\mathbf{B}(\mathcal{T})_Y = \{\, A \cap Y \mid A \in \mathbf{B}(\mathcal{T}) \,\}$. Equivalently, we have that $\mathbf{B}(\mathcal{T})_Y = \mathbf{B}(\mathcal{T}_Y)$.

Theorem 6.5 (The Borel Schröder-Bernstein Theorem)**.** *Suppose that X, Y are Polish spaces, that $f : X \to Y$ is a Borel isomorphism between X and $f(X)$ and that $g : Y \to X$ is a Borel isomorphism between Y and $g(Y)$. Then there exists a Borel isomorphism $h : X \to Y$.*

Proof. We follow the standard proof of the Schröder-Bernstein Theorem, checking that all of the sets and functions involved are Borel. Define inductively

$$X_0 = X \qquad\qquad Y_0 = Y$$

$$X_{n+1} = g(f(X_n)) \qquad\qquad Y_{n+1} = f(g(Y_n))$$

Then an easy induction shows that X_n, Y_n, $f(X_n)$ and $g(Y_n)$ are Borel for each $n \in \mathbb{N}$. Hence $X_\infty = \bigcap X_n$ and $Y_\infty = \bigcap Y_n$ are also Borel. Furthermore, we have that

$$f(X_n \smallsetminus g(Y_n)) = f(X_n) \smallsetminus Y_{n+1}$$

$$g(Y_n \smallsetminus f(X_n)) = g(Y_n) \smallsetminus X_{n+1}$$

$$f(X_\infty) = Y_\infty$$

Finally define

$$A = X_\infty \cup \bigcup_n (X_n \smallsetminus g(Y_n))$$

$$B = \bigcup_n (Y_n \smallsetminus f(X_n))$$

Then A, B are Borel, $f(A) = Y \smallsetminus B$ and $g(B) = X \smallsetminus A$. Thus we can define a Borel bijection $h : X \to Y$ by

$$h(x) = \begin{cases} f(x), & \text{if } x \in A; \\ g^{-1}(x) & \text{otherwise.} \end{cases}$$

□

Definition 6.6. A Hausdorff topological space X is *zero-dimensional* iff X has a basis consisting of clopen sets.

Theorem 6.7. *Every zero-dimensional Polish space X can be embedded in the Cantor set $2^{\mathbb{N}}$.*

Proof. Fix a countable basis $\{U_n\}$ of clopen sets and define $f : X \to 2^{\mathbb{N}}$ by

$$f(x) = (\,\chi_{U_0}(x), \cdots, \chi_{U_n}(x), \cdots),$$

where $\chi_{U_n} : X_n \to 2$ is the characteristic function of U_n. Since the characteristic function of a clopen set is continuous, it follows that f is continuous; and since $\{U_n\}$ is a basis, it follows that f is an injection. Also

$$f(U_n) = f(X) \cap \{\varphi \in 2^{\mathbb{N}} \mid \varphi(n) = 1\}$$

is open in $f(X)$. Hence f is an embedding. □

Thus Theorem 6.2 is an immediate consequence of Theorem 6.5, Corollary 3.13 and the following result.

Theorem 6.8. *Let $(X, \mathcal{T})$ be a Polish space. Then there exists a Borel isomorphism $f : X \to 2^{\mathbb{N}}$ between X and $f(X)$.*

Proof. Let $\{U_n\}$ be a countable basis of open sets of $(X, \mathcal{T})$ and let $F_n = X \smallsetminus U_n$. By Lemma 5.4, for each $n \in \mathbb{N}$, the topology generated by $\mathcal{T} \cup \{F_n\}$ is Polish. Hence, by Lemma 5.5, the topology $\mathcal{T}'$ generated by $\mathcal{T} \cup \{F_n \mid n \in \mathbb{N}\}$ is Polish. Clearly the sets of the form

$$U_n \cap F_{m_1} \cap \cdots \cap F_{m_t}$$

form a clopen basis of $(X, \mathcal{T}')$. Hence, applying Theorem 6.7, there exists an embedding $f : (X, \mathcal{T}') \to 2^{\mathbb{N}}$. Clearly $f : (X, \mathcal{T}) \to 2^{\mathbb{N}}$ is a Borel isomorphism between X and $f(X)$. □

7. The nonexistence of a well-ordering of $\mathbb{R}$

Theorem 7.1. *There does not exists a Borel well-ordering of* $2^{\mathbb{N}}$.

Corollary 7.2. *There does not exists a Borel well-ordering of* $\mathbb{R}$.

Proof. An immediate consequence of Theorems 7.1 and 6.2. □

Definition 7.3. The *Vitali equivalence relation* E_0 on $2^{\mathbb{N}}$ is defined by:

$$(a_n)\, E_0\, (b_n) \quad \text{iff} \quad \text{there exists } m \text{ such that } a_n = b_n \text{ for all } n \geq m.$$

Definition 7.4. If E is an equivalence relation on X, then an *E-transversal* is a subset $T \subseteq X$ which intersects every E-class in a unique point.

Theorem 7.5. *There does not admit a Borel E_0-transversal.*

Let $C_2 = \{0,1\}$ be the cyclic group of order 2. Then we can regard $2^{\mathbb{N}} = \prod_n C_2$ as a direct product of countably many copies of C_2. Define

$$\Gamma = \bigoplus_n C_2 = \{(a_n) \in \prod_n C_2 \mid a_n = 0 \text{ for all but finitely many } n\}.$$

Then Γ is a subgroup of $\prod_n C_2$ and clearly

$$(a_n)\, E_0\, (b_n) \quad \text{iff} \quad (\exists \gamma \in \Gamma) \;\; \gamma \cdot (a_n) = (b_n).$$

Definition 7.6. A *probability measure* μ on an algebra $\mathcal{B} \subseteq \mathcal{P}(X)$ of sets is a function $\mu : \mathcal{F} \to [0,1]$ such that:

(i) $\mu(\emptyset) = 0$ and $\mu(X) = 1$.

(ii) If $A_n \in \mathcal{B}$, $n \in \mathbb{N}$, are pairwise disjoint and $\bigcup A_n \in \mathcal{B}$, then

$$\mu(\bigcup A_n) = \sum \mu(A_n).$$

Example 7.7. Let $\mathcal{B}_0 \subseteq 2^{\mathbb{N}}$ consist of the clopen sets of the form

$$A_{\mathcal{F}} = \{(a_n) \mid (a_0, \cdots, a_{m-1}) \in \mathcal{F}\},$$

where $\mathcal{F} \subseteq 2^m$ for some $m \in \mathbb{N}$. Then $\mu(A_{\mathcal{F}}) = |\mathcal{F}|/2^m$ is a probability measure on $\mathcal{B}_0$. Furthermore, it is easily checked that μ is Γ-*invariant* in the sense that $\mu(\gamma \cdot A_{\mathcal{F}}) = \mu(A_{\mathcal{F}})$ for all $\gamma \in \Gamma$.

Theorem 7.8. *μ extends to a Γ-invariant probability measure on* $\mathbf{B}(2^{\mathbb{N}})$.

Sketch Proof. First we extend μ to arbitrary open sets U by defining

$$\mu(U) = \sup\{\mu(A) \mid A \in \mathcal{B}_0 \text{ and } A \subseteq U\}.$$

Then we define an *outer measure* μ^* on $\mathcal{P}(2^{\mathbb{N}})$ by setting

$$\mu^*(Z) = \inf\{\mu(U) \mid U \text{ open and } Z \subseteq U\}.$$

Unfortunately there is no reason to suppose that μ^* is countably additive; and so we should restrict μ^* to a suitable subcollection of $\mathcal{P}(2^{\mathbb{N}})$. A minimal requirement for Z to be a member of this subcollection is that

$$(\dagger) \qquad \mu^*(Z) + \mu^*(2^{\mathbb{N}} \smallsetminus Z) = 1;$$

and it turns out that:

(i) μ^* is countably additive on the collection $\mathcal{B}$ of sets satisfying condition ($\dagger$).
(ii) $\mathcal{B}$ is a σ-algebra contain the open subsets of $2^{\mathbb{N}}$.
(iii) If $U \in \mathcal{B}$ is open, then $\mu^*(U) = \mu(U)$.

Clearly μ^* is Γ-invariant and hence the probability measure $\mu^* \restriction \mathbf{B}(2^{\mathbb{N}})$ satisfies our requirements. □

Remark 7.9. In order to make the proof go through, it turns out to be necessary to define $\mathcal{B}$ to consist of the sets Z which satisfy the apparently stronger condition that

$$(\dagger\dagger) \qquad \mu^*(E \cap Z) + \mu^*(E \smallsetminus Z) = \mu^*(E) \qquad \text{for } \textit{every } E \subseteq 2^{\mathbb{N}}.$$

Proof of Theorem 7.5. If T is a Borel tranversal, then T is μ-measurable. Since

$$2^{\mathbb{N}} = \bigsqcup_{\gamma \in \Gamma} \gamma \cdot T,$$

it follows that

$$1 = \mu(2^{\mathbb{N}}) = \sum_{\gamma \in \Gamma} \mu(\gamma \cdot T).$$

But this is impossible, since $\mu(\gamma \cdot T) = \mu(T)$ for all $\gamma \in \Gamma$. □

We are now ready to present the proof of Theorem 7.1. Suppose that $R \subseteq 2^{\mathbb{N}} \times 2^{\mathbb{N}}$ is a Borel well-ordering of $2^{\mathbb{N}}$ and let E_0 be the Vitali equivalence relation on $2^{\mathbb{N}}$. Applying Theorem 7.5, the following claim gives the desired contradiction.

Claim 7.10. *$T = \{ x \in 2^{\mathbb{N}} \mid x$ is the R-least element of $[x]_{E_0} \}$ is a Borel E_0-transversal.*

Proof of Claim 7.10. Clearly T is an E_0-transversal and so it is enough to check that T is Borel. If $\gamma \in \Gamma$, then the map $x \mapsto \gamma \cdot x$ is a homeomorphism and it follows easily that

$$M_\gamma = \{(x, \gamma \cdot x) \mid x \in 2^{\mathbb{N}}\}$$

is a closed subset of $2^{\mathbb{N}} \times 2^{\mathbb{N}}$. Hence

$$L_\gamma = \{(x, \gamma \cdot x) \in \mid x \, R \, \gamma \cdot x\} = M_\gamma \cap R$$

is a Borel subset of $2^{\mathbb{N}} \times 2^{\mathbb{N}}$. Let $f_\gamma : 2^{\mathbb{N}} \to 2^{\mathbb{N}} \times 2^{\mathbb{N}}$ be the continuous map defined by $f_\gamma(x) = (x, \gamma \cdot x)$. Then

$$T_\gamma = \{x \in 2^{\mathbb{N}} \mid x \, R \, \gamma \cdot x\} = f_\gamma^{-1}(L_\gamma)$$

is a Borel subset of $2^{\mathbb{N}}$ and hence $T = \bigcap_{\gamma \neq 0} T_\gamma$ is also Borel. □